

LESSON 40 *Newton and Leibniz · The differential*

40.A

Newton and Leibniz

We have defined the derivative of a function to be the limit of the ratio of the change in y to the change in x as the change in x approaches zero.

$$\frac{d}{dx} f(x) = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Sir Isaac Newton, an Englishman, and Gottfried Wilhelm Leibniz, a German, invented calculus independently of one another in the seventeenth century. To designate the derivative of a function, Newton placed a dot over the y and wrote \dot{y} (read “y dot”). Over the years the dot has changed to a prime and now we write y' (read “y prime”) as Newton’s designation of the derivative. The derivative as conceived by Newton and indicated by his notation \dot{y} was a single entity and had no numerator or denominator.

Leibniz designated the derivative with the fractional notation $\frac{dy}{dx}$ and considered the derivative to be a fraction of very small quantities dy and dx which he called *infinitesimals* and which he moved about by using the rules of algebra. Leibniz could multiply $\frac{dy}{dx}$ by dx and get dy by canceling the dx above and the dx below as we show here.

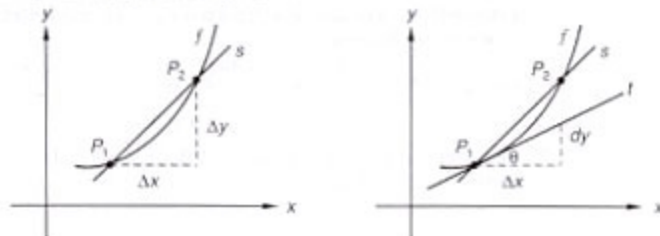
$$\frac{dy}{dx} dx = dy$$

Many scientists prefer the notation of Leibniz because it facilitates the solution of practical problems whose solutions would be more difficult using the notation of Newton. In this book we will use both notations. We will use the notation of Leibniz often because this notation is easier for the beginner to understand. It is easy to think of $\frac{dy}{dx}$ as meaning almost the same thing as a very, very small Δy over a very, very small Δx . In the next section we will show that dy can have a meaning all by itself and dx can have a meaning all by itself; so this thought process is acceptable.

40.B

The differential

The slope of the graph of a function at a point has the same value as the derivative of the function at that point. In the figure on the left, if we begin at P_1 and move to P_2 , the change in x is Δx and the change in y is Δy . We would like to show that in the limit, as P_2 moves closer to P_1 , the Δy in the left-hand figure is approximately equal to the dy in the right-hand figure.



In the right-hand figure on page 205, a segment of the tangent to the curve at P_1 forms the hypotenuse of a right triangle whose vertical side is dy . If P_2 moves down the curve and gets closer to P_1 , the secant s will get closer to the tangent t and the length of the segment dy in the right-hand figure will get closer to the length of the whole side which we call Δy . We call the segment dy the *differential of the function*. We can solve the small triangle for dy by using the definition of $\tan \theta$.

$$\frac{dy}{\Delta x} = \tan \theta \quad \rightarrow \quad dy = \tan \theta \Delta x$$

But the tangent of θ , the slope of the tangent line, equals the derivative of the function f , which is $f'(x)$. If we replace $\tan \theta$ with $f'(x)$, we have

$$dy = f'(x) \Delta x$$

We can show that in the limit, the value of Δy equals dy , which is the differential. To do this, we show that the ratio of Δy to dy approaches 1 as Δx approaches 0.

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{dy} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{f'(x) \Delta x} \quad \left(\frac{\Delta y}{\Delta x} \text{ definition of } dy \right)$$

Next we divide the numerator and the denominator by Δx . Now we have

$$\lim_{\Delta x \rightarrow 0} \frac{\frac{\Delta y}{\Delta x}}{f'(x)}$$

The denominator, $f'(x)$, has already been determined and is considered to be a constant and does not change as Δx changes. The limit of the numerator $\Delta y/\Delta x$ as Δx approaches zero is the derivative, which is also $f'(x)$, so we have

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{dy} = \frac{f'(x)}{f'(x)} = 1$$

We have shown that in the limit, dy is approximately equal to Δy because the ratio Δy over dy approaches 1. Thus, $dy \approx \Delta y$ if $dx = \Delta x$ and if Δx is "small." So we may write

$$dy = f'(x) dx$$

We have shown that it is reasonable to consider dy and dx as variables that represent very small quantities. If we do this, dy and dx can be multiplied and divided in the same manner as other variables. If we divide both sides of this last expression by dx , we get

$$\frac{dy}{dx} = f'(x)$$

The derivative of a function has meaning as the rate of change of the function or as the slope of the graph of the function. The differential has no similar meaning. We use the differential because its use will permit algebraic manipulations that are helpful in the solutions of problems.

We use a small letter d as an operational indicator to indicate the operation of taking a differential. If we have the functions

$$y = 5x^4 \quad f(t) = 3 \sin t \quad g(u) = \ln u$$

we could indicate the operation of finding the differentials of these functions by writing the following:

$$dy \quad df \quad dg$$

The differential of a function of x is the derivative with respect to x with dx as an additional factor. The differential of a function of t is the derivative of the function with respect to t with dt as an additional factor. The differential of a function of u is the derivative of the function with respect to u with du as an additional factor. Thus

$$dy = 20x^3 dx \quad df = 3 \cos t dt \quad dg = \frac{1}{u} du = \frac{du}{u}$$

The differential of an algebraic sum of functions is the sum of the individual differentials.

Example 40.1 If $y = 3x^{-2} + 2 \cos u - 5e^t$, find dy .

Solution The differential of a sum is the sum of the differentials. Thus

$$dy = -6x^{-3} dx - 2 \sin u du - 5e^t dt$$

Note that we did not find the differential of y with respect to a particular variable. The variable in each of the functions was different, and the differential of y is the sum of the differentials of the individual terms.

Problem set 40

1. It was 1 o'clock when the bell in the clock tower chimed once. In how many minutes would the hands of the clock be pointing in precisely the same direction?
2. At noon, the bacteria colony covered 20 square centimeters. The area which the bacteria covered was found to increase exponentially. If at 2 p.m. the bacteria covered 50 square centimeters, how much area was covered at 5 p.m.?

Compute dy in Problems 3–5.

$$3. y = \frac{3}{x^2} + 2 \sin u + 2e^t \qquad 4. y = 2 \ln |u| - \frac{4}{\sqrt{x}}$$

$$5. y = \sqrt[3]{t} + 2$$

$$6. \text{ Sketch the graph of } y = \frac{(x-3)(x+2)}{x(x-1)(x+1)}.$$

7. Write the equation of the line which can be drawn tangent to the graph of $y = \sqrt{x}$ at $x = 4$.

Approximate to two decimal places the answers to Problems 8 and 9.

$$8. \text{ Find } \left. \frac{d^3y}{dx^3} \right|_{x=3} \text{ if } y = \sin x.$$

$$9. \text{ Find } \left. \frac{du}{dx} \right|_{x=1} \text{ if } u = 4 \ln |x| + 2e^x - \cos x.$$

10. Write the key identities for practice, and then develop an identity for $\tan 2A$. Then, if $\tan A = -\frac{1}{4}$, find the value of $\tan 2A$.
11. Find the volume of a prism whose length is L centimeters, each of whose ends are equilateral triangles, each of whose sides are E centimeters long.
12. Express the four fourth roots of $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$ in polar form.
13. Find all values of x such that $0 \leq x < 2\pi$ and $\sin 4x = -\frac{1}{2}$.